1. A psychological study of memory randomly assigned six subjects to one of three combinations of study time (1 minute or 5 minutes) and refresher (present or absent). The three treatments had the following structure:

	Refresher	
Study time	Absent	Present
1 minute	μ_{1A}	μ_{1P}
5 minutes	μ_{5A}	

The 5 minutes study time, with refresher, treatment was not used in the study. There are two subjects for each of the three treatments that were used. The investigators propose to use a non-full rank additive factor effects model to analyze the data: $Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$.

(a) Write out the X matrix for this model with 6 subjects.

(b) Is the main effect of refresher, i.e. the average difference between refresher present and refresher absent, estimable? Briefly explain why or why not.

(c) Is the response mean for the 5 minute study time, with refresher, treatment estimable? Briefly explain why or why not.

2. An observational study evaluated the association between 7 categorical factors and the number of hours spent playing video games in a week. We consider only data from the first 15 subjects. Each categorical factor has two levels and is coded as one column containing either 0 or 1. The $\boldsymbol{X}$ matrix for those subjects is:

1	0	0	1	1	0	0	0
1	1	1	0	0	1	1	0
1	1	1	0	0	1	0	0
1	1	0	0	0	1	1	1
1	1	0	1	1	0	1	0
1	0	1	0	0	1	1	0
1	0	1	0	0	1	0	0
1	0	0	1	0	0	1	1
1	0	1	1	0	0	1	0
1	0	1	0	0	1	0	0
1	0	0	1	1	0	0	0
1	0	0	1	1	0	0	0
1	1	1	0	0	1	0	0
1	1	1	0	0	1	0	0
1	1	0	1	1	0	0	0

Some additional, potentially useful, information:

The Normal Gauss-Markov model $Y = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$, $\boldsymbol{\epsilon} \sim N(0, \sigma^2)$, was fit to the data.

The eigen decomposition of $\boldsymbol{P}_X$, i.e. $\boldsymbol{UDU}' = P_X$ is

$$\boldsymbol{U} = \begin{bmatrix}
-0.133 & -0.089 & 0.215 & 0.432 & -0.043 & 0.019 & 0 & 0.859 & 0 & 0 & 0 & 0 & -0.045 & 0 & 0.002 \\
0.35 & 0.292 & -0.042 & -0.047 & -0.154 & 0.343 & -0.199 & 0.119 & 0.449 & -0.158 & 0.106 & -0.206 & 0.288 & -0.121 & -0.464 \\
0.379 & 0.01 & 0.053 & 0.047 & 0.334 & 0.024 & -0.466 & 0.054 & -0.291 & 0.191 & -0.187 & 0.509 & 0.301 & 0.102 & -0.052 \\
0.284 & -0.542 & 0.226 & -0.247 & -0.248 & 0.398 & -0.094 & 0.024 & 0.281 & 0.023 & 0.007 & 0.163 & -0.193 & -0.01 & 0.376 \\
-0.105 & 0.281 & 0.12 & 0.118 & -0.079 & 0.635 & -0.104 & -0.106 & -0.42 & 0.295 & -0.209 & -0.285 & -0.214 & -0.112 & 0.056 \\
0.293 & 0.205 & -0.041 & 0.174 & -0.605 & 0.046 & 0.396 & -0.036 & -0.311 & -0.161 & 0.096 & 0.329 & 0.119 & 0.243 & 0.032 \\
0.322 & -0.077 & 0.054 & 0.268 & -0.118 & -0.273 & 0.069 & -0.1 & 0.224 & 0.487 & -0.336 & -0.41 & 0.107 & 0.357 & 0.075 \\
-0.025 & -0.658 & -0.456 & 0.143 & -0.001 & 0.207 & 0.094 & -0.024 & -0.281 & -0.023 & -0.007 & -0.163 & 0.193 & 0.01 & -0.376 \\
0.041 & 0.176 & -0.724 & 0.343 & 0.093 & 0.152 & -0.094 & 0.024 & 0.281 & 0.023 & 0.007 & 0.163 & -0.193 & -0.01 & 0.376 \\
0.322 & -0.077 & 0.054 & 0.268 & -0.118 & -0.273 & -0.173 & -0.1 & -0.215 & -0.142 & 0.149 & -0.247 & 0.117 & -0.668 & 0.279 \\
-0.133 & -0.089 & 0.215 & 0.432 & -0.043 & 0.019 & -0.067 & -0.311 & 0.195 & -0.438 & -0.554 & 0.193 & -0.149 & -0.07 & -0.19 \\
-0.133 & -0.089 & 0.215 & 0.432 & -0.043 & 0.019 & -0.225 & -0.311 & 0.106 & 0.254 & 0.646 & 0.135 & -0.149 & 0.138 & -0.198 \\
0.379 & 0.01 & 0.053 & 0.047 & 0.334 & 0.024 & -0.078 & 0.02 & -0.201 & -0.487 & 0.165 & -0.327 & -0.361 & 0.435 & 0.045 \\
0.379 & 0.01 & 0.053 & 0.047 & 0.334 & 0.024 & 0.544 & 0.02 & 0.063 & 0.246 & 0 & 0.189 & -0.378 & -0.339 & -0.291 \\
-0.076 & -0.001 & 0.215 & 0.212 & 0.408 & 0.317 & 0.396 & -0.131 & 0.118 & -0.111 & 0.118 & -0.043 & 0.557 & 0.044 & 0.33
\end{bmatrix}$$

$$\boldsymbol{D} = diag\{[1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0]'\}$$

The $(\boldsymbol{X}'\boldsymbol{X})^-$ matrix is:

$$\begin{bmatrix}
0.075 & -0.073 & -0.025 & 0.025 & -0.014 & 0.050 & -0.099 & 0.114 \\
-0.073 & 0.352 & 0.074 & 0.110 & -0.162 & -0.183 & -0.082 & 0.015 \\
-0.025 & 0.074 & 0.263 & 0.179 & -0.224 & -0.204 & -0.072 & -0.064 \\
0.025 & 0.110 & 0.179 & 0.271 & -0.286 & -0.246 & -0.178 & 0.131 \\
-0.014 & -0.162 & -0.224 & -0.286 & 0.466 & 0.271 & 0.222 & -0.256 \\
0.050 & -0.183 & -0.204 & -0.246 & 0.271 & 0.295 & 0.080 & -0.017 \\
-0.099 & -0.082 & -0.072 & -0.178 & 0.222 & 0.080 & 0.437 & -0.249 \\
0.114 & 0.015 & -0.064 & 0.131 & -0.256 & -0.017 & -0.249 & 0.434
\end{bmatrix}$$

The $\hat{\beta}$ vector is:

$$[1.4, 0.6, -0.32, 0.54, -0.03, 0.23, -0.49, 0.01]'$$

The estimated error variance, $\hat{\sigma}^2 = 2.3$.

(a) What is the dimension (# rows, # columns) of the projection matrix, $\boldsymbol{P}_X$, derived from $\boldsymbol{X}$?

(b) The investigators are interested in the effect of variable X1, i.e. the variable in the second column of $\boldsymbol{X}$. β_1, the β for this variable is estimable. Estimate β_1

(c) Estimate Var β_1.

(d) Test Ho: $\beta_1 = 0$. Report an appropriate test statistic and state its distribution under Ho. You do not need to calculate the p-value.

(e) The investigators are interested in the power of this test of $\beta_1 = 0$. Calculate the non-centrality parameter for this test, assuming that $\beta_1 = 1.0$, $\sigma^2 = 2.3$, and 15 subjects with values of the various factors given by the observed $\boldsymbol{X}$ matrix.

(f) The investigators are also interested in $\beta_3 - \beta_5$. Estimate $\beta_3 - \beta_5$.

(g) Estimate Var $\beta_3 - \beta_5$.

(h) How many degrees of freedom are associated with the estimated error variance? I.e. what is the error d.f.?

Briefly explain (1 sentence) how you determined your answer.

(i) The investigators are primarily interested in three tests, Ho: $\beta_1 = 0$, Ho: $\beta_3 - \beta_5 = 0$, and Ho: $\beta_6 = 0$. If the important difference is 1.2 in each of the three tests, which test will have the higher power?

Briefly explain your answer.

3. Meat is aged to increase its tenderness. Data were collected from 2 animals. For each animal, the muscle of interest was cut into 8 steaks. Two steaks from each animal were aged for 1 day, 3 days, 7 days, or 14 days. Aging time was randomly assigned to steaks within animal. Tenderness was measured three times on each steak and averaged to get a single estimate of tenderness for each of the 16 steaks. Previous data suggests that the variance of tenderness within aging treatment and animal increases with aging. The variance among steaks aged for x days is x times the variance of steaks aged for 1 day. Errors are assumed to be independent.

(a) Using factor effects notation, write an appropriate model based on the information provided above. The investigators do not want you to assume anything not based on the information provided above. Use subscripts instead of matrices in your model. Define the terms in your model and your subscripts.

(b) What name is commonly given to a model like the one you wrote in part 3a?

(c) The first four observations from animal 1 are:

Animal	Aging Time
1	1
1	3
1	7
1	14

Write out the 4×4 block of Σ, the variance-covariance matrix of the errors corresponding to these observations.

I fit a model using an $\boldsymbol{X}$ matrix with sum-to-zero coding for the block effects and orthogonal polynomials to represent the effects of aging. That $\boldsymbol{X}$ matrix is

$$\begin{bmatrix} X0 & B1 & X1 & X2 & X3 \\ 1 & 1 & -1.585 & 5.423 & -9.284 \\ 1 & 1 & 0.415 & -12.548 & 49.373 \\ 1 & 1 & 4.415 & -24.489 & -60.345 \\ 1 & 1 & 11.415 & 31.614 & 20.256 \\ 1 & -1 & -1.585 & 5.423 & -9.284 \\ 1 & -1 & 0.415 & -12.548 & 49.373 \\ 1 & -1 & 4.415 & -24.489 & -60.345 \\ 1 & -1 & 11.415 & 31.614 & 20.256 \\ 1 & 1 & -1.585 & 5.423 & -9.284 \\ 1 & 1 & 0.415 & -12.548 & 49.373 \\ 1 & 1 & 4.415 & -24.489 & -60.345 \\ 1 & 1 & 11.415 & 31.614 & 20.256 \\ 1 & -1 & -1.585 & 5.423 & -9.284 \\ 1 & -1 & 0.415 & -12.548 & 49.373 \\ 1 & -1 & 4.415 & -24.489 & -60.345 \\ 1 & -1 & 11.415 & 31.614 & 20.256 \end{bmatrix}$$

X1 is the column of coefficients for the linear orthogonal polynomial, X2 is the column for the quadratic orthogonal polynomial, and X3 is the column for the cubic orthogonal polynomial. The $\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X}$ matrix is:

$$\begin{bmatrix} 6.19 & 0 & 0 & 0 & 0 \\ 0 & 6.19 & 0 & 0 & 0 \\ 0 & 0 & 58.646 & 0 & 0 \\ 0 & 0 & 0 & 955.803 & 0 \\ 0 & 0 & 0 & 0 & 5793.105 \end{bmatrix}$$

Sequential (Type I) SS for the model sequence: X0, X0+B1, X0+B1+X1, X0+B1+X1+X2, X0+B1+X1+X2+X3 are given in the attached R and SAS output.

(c) Each sequential SS represents a comparison between a full and a reduced model. What pair of models is being compared in the computation of the SS labelled X2 on the output?

(d) Calculate the F statistic that tests the null hypothesis of no effect of aging, i.e. that all four aging times have the same mean. If this is not possible with output provided, what additional information do you need?

(e) What are the numerator and denominator d.f. for the F statistic in part 3d?

(f) Estimate the error variance for an observation at day 3.

(g) Partial (Type III) SS also represent comparisons between a pair of models. What pair of models would be compared by the partial SS for X2?

(h) Calculate the partial (Type III) SS associated with the test of X2 = 0. If this is not possible with output provided, what additional information do you need?

Computer output for problem 3, meat aging

R output:

```
Analysis of Variance Table

Response: y
          Df  Sum Sq Mean Sq F value    Pr(>F)
B1         1 0.77588 0.77588 14.0277 0.0032359 **
X1         1 1.11516 1.11516 20.1618 0.0009159 ***
X2         1 0.12157 0.12157  2.1980 0.1662616
X3         1 0.06819 0.06819  1.2328 0.2905361
Residuals 11 0.60841 0.05531
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

SAS output

```
                             Sum of
Source            DF        Squares Mean Square  F Value    Pr > F

Model              4     2.08079355  0.52019839     9.41    0.0015

Error             11     0.60841383  0.05531035

Corrected Total   15     2.68920737


 R-Square     Coeff Var      Root MSE        y Mean

 0.773757      3.357315      0.235182      7.005047


Source   DF    Type I SS   Mean Square  F Value    Pr > F

b1        1   0.77587544    0.77587544    14.03    0.0032
x1        1   1.11515582    1.11515582    20.16    0.0009
x2        1   0.12157463    0.12157463     2.20    0.1663
x3        1   0.06818766    0.06818766     1.23    0.2905
```